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The force of attraction between two solids with different temperatures

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Abstract. The methods of fluctuation electrodynamics and the molecular cohesive force theory have been used to obtain an expression for the attractive force density between two absorbing media varying in temperature and separated by a nonabsorbing plane-parallel layer. The spectral density of the attractive force was calculated as the stress tensor projection on the exterior normal to the solid surface. The mean-square characteristics of the fluctuation field of these media were sought using the generalized Kirchhoff's law and the Green function of the related regular problem. A variety of solutions have been obtained depending on a temperature relationship of the interacting solid media. It is shown that for solids of the same temperature the obtained expression yields a formula describing the force of attraction in equilibrium.

1. Introduction

A theory of the interacting force between two atoms without a constant dipole moment was developed in [1, 2]. The existence of the attractive forces between neutral particles determines existence of such forces also between two solids separated by a very small gap. A macroscopic theory of molecular forces of attraction between solids separated by a vacuum space or a nonabsorbing medium was proposed in [3]. The basic idea of this theory is that media interact via a fluctuation electromagnetic field that is always present inside any absorbing medium and emerges from the latter as thermal radiation and a near-quasistationary (evanescent) field. The author of the theory has considered interaction of solid media filling semi-infinite solids with plane-parallel boundaries, separated by some gap, and calculated spectral density of the cohesive force by solving a boundary-value problem with the relevant boundary conditions and random extraneous sources distributed in these media. The solution has a broad generality and is applicable to any media at any temperature. Besides, in the limiting case of a rarified medium it yields the well known laws of interaction for individual atoms. The same result was obtained in the monograph [4] where the generalized Kirchhoff's formula, the complex Lorentz lemma, and the Green function of the regular diffraction problem for a point source located in a plane nonabsorbing layer between two absorbing homogeneous isotropic media were applied to the problem of interest. In [5] the authors provide a general theory of the van der Waals forces, constructed using the methods of the quantum theory of field, which can be used for the determination of the forces between two solids separated by an absorbing layer. It should be noted that the formulae obtained in [3] are only slightly changed here. Interestingly, both attraction and

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repulsion processes are possible depending on a relationship between the dielectric constants of the three interacting media. Note that the focus in the above studies is on the equilibrium situation, when all media involved in the force interaction have equal temperatures.

First experimental measurements of the attractive force between two plane-parallel plates separated by a layer about 10^3 Å thick [6] were in good agreement with the developed theory. Further attempts at experimental verification of the theory by different authors [7–9] also showed good qualitative and quantitative agreement between theory and experiment.

The latest advancement in probe microscopy and physics of microcontacts has imparted new meaning and stimulus to the studies of interaction of solids via thermal fluctuation fields. In particular, extensive application of probe microscopes to investigations and local modification of surface properties in condensed matter [10, 11] gave rise to problems in which one has to consider ponderomotive interaction and energy exchange between sample and tip that differ in temperature.

In this work, an expression for the attractive force density between two arbitrary absorbing media separated by a nonabsorbing plane-parallel layer and heated to different temperatures was obtained by calculating the spectral component of the Maxwell stress tensor. The limiting cases, when one medium has a much higher temperature than the other, are considered. It is also shown that, if media are of the same temperature, the attractive force expression is identical to the formulae for the equilibrium case.

2. Problem statement

The electrodynamical theory of thermal fluctuations provides a means for a detailed analysis of the structure and properties of the fluctuating electromagnetic fields of heated solids. These properties include, for example, spectral intensities of electric and magnetic energy, Poynting vector, Maxwell stress tensor. The relevant statistic mean square values of the bilinear forms, i.e. various correlation functions or spectral intensities of fluctuations can be found by solving an ordinary boundary problem of electrodynamics with the use of the electrodynamical fluctuation–dissipation theorem (FDT) for the correlation functions of extraneous fluctuating currents distributed over absorbing medium.

The second moments of the spectral amplitude of a fluctuating field can be found in another way, using their relationship with thermal losses of the diffraction field of point sources from the formulae generalizing the classical Kirchhoff's equation [4]:

$$\pm\langle A_{\ell 1}(\boldsymbol{r}_1)B^*_{\ell 2}(\boldsymbol{r}_2)\rangle = \frac{2}{\pi}\Theta(\omega, T)Q_{AB^*}(\boldsymbol{\ell}_2 1, \boldsymbol{r}_1; \boldsymbol{\ell}_2, \boldsymbol{r}_2) \tag{1}$$

where $A_{\ell 1}$ and $B_{\ell 2}$ define the two out of six chosen components of strength $\boldsymbol{E}$, $\boldsymbol{H}$ of the thermal field in the $\boldsymbol{\ell}_2 1$ and $\boldsymbol{\ell}_2$ orientations of point sources; $\Theta(\omega, T) = (\hbar\omega/2)\coth(\hbar\omega/2k_B T)$ is the average energy of oscillator at temperature T, k_B is Boltzmann's constant, Q_{AB^*} are the mixed thermal losses incurred in the investigated material by the diffraction field from point sources placed at points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The plus sign corresponds to two electric or two magnetic components, the minus sign is for an electric- and a magnetic-component. Complex conjugation is designated by the asterisk, the angle brackets indicate an average over an ensemble of realizations of the random sources. It is obvious that a search for the Green function of the regular problem can be a simpler process in some cases.

If the second moments of the spectral amplitudes of a fluctuating field between two solids are known, we find the density of the force acting on these solids by frequency-integrating the spectral component of the Maxwell stress tensor projection onto the direction of the unit normal to the given surface. The spectral density of this tensor across positive frequencies in some point $\boldsymbol{r}$ [4], is

$$T_\omega^{\alpha\beta}(\boldsymbol{r}) = \frac{1}{8\pi}\{2\langle E_\alpha(\omega,\boldsymbol{r})D_\beta^*(\omega,\boldsymbol{r})\rangle - \langle \boldsymbol{E}(\omega,\boldsymbol{r})\boldsymbol{D}^*(\omega,\boldsymbol{r})\rangle\delta_{\alpha\beta} + \text{c.c.}\}$$
$$+\frac{1}{8\pi}\{2\langle H_\alpha(\omega,\boldsymbol{r})B_\beta^*(\omega,\boldsymbol{r})\rangle - \langle \boldsymbol{H}(\omega,\boldsymbol{r})\boldsymbol{B}^*(\omega,\boldsymbol{r})\rangle\delta_{\alpha\beta} + \text{c.c.}\} \qquad (2)$$

where the expansion

$$\boldsymbol{E}(t,\boldsymbol{r}) = \int_{-\infty}^{\infty}\boldsymbol{E}(\omega,\boldsymbol{r})\exp(\mathrm{i}\omega t)\,\mathrm{d}\omega$$

is used for the real stationary fields and, similarly, for the field $\boldsymbol{H}$ and inductions $\boldsymbol{D}$ and $\boldsymbol{B}$.

We now seek the density of the force between two absorbing homogeneous isotropic media with different temperatures (figure 1). Assume the half-space $z \leqslant 0$ filled with a material characterized by complex dielectric constants ε_1, μ_1, the half-space $z \geqslant l$ filled with a medium with the constants ε_2, μ_2, while the gap between them filled with a nonabsorbing medium with the real dielectric constants ε, μ. The spectral density of the force will be sought as the zz-component of the Maxwell stress tensor (2) on the surface $z = 0$. We assume that two independent systems of extraneous random sources of fluctuation fields are located in two thermostats with temperatures T_1 and T_2 kept constant. Using the reciprocity theorem and the principle of superposition of fields, we make the required square combinations from the vector components of the fluctuation field. Next, after averaging over the equilibrium ensembles of random currents, we apply the electrodynamic FDT to obtain mean square characteristics of the fluctuation field, expressed via the thermal losses of diffraction fields in either medium, induced by point dipoles $\boldsymbol{p}$ placed in the gap at some distance h from the lower medium and oriented in an appropriate fashion. Thus, multiplying the losses $Q^{(2)}$ in the first medium by $\Theta(\omega, T_1)$, and in the second medium-$Q^{(2)}$ by $\Theta(\omega, T_2)$, we have:

$$\langle|E_z(\boldsymbol{r})|^2\rangle = \frac{2}{\pi}\{\Theta(\omega,T_1)Q_{ee^*}^{(1)}(\ell_z;\boldsymbol{r}) + \Theta(\omega,T_2)Q_{ee^*}^{(2)}(\ell_z;\boldsymbol{r})\}$$
$$\langle|H_z(\boldsymbol{r})|^2\rangle = \frac{2}{\pi}\{\Theta(\omega,T_1)Q_{mm^*}^{(1)}(\ell_z;\boldsymbol{r}) + \Theta(\omega,T_2)Q_{mm^*}^{(2)}(\ell_z;\boldsymbol{r})\} \qquad (3)$$

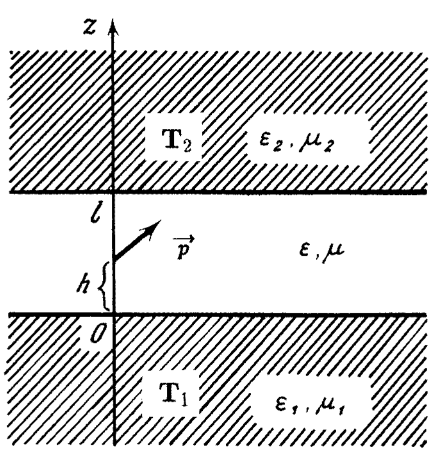

Figure 1. The model of interacting solids.

where notations ee^* and mm^* indicate the requirement to find the losses of regular fields induced by electric and magnetic point dipoles, respectively.

In a similar way, we need to find other mean square values of the fluctuation field components in the $0x$ and $0y$ and, using (2), derive the sought-after expression for the spectral density of the force $\widetilde{F}_\omega = T_\omega^{zz}$. Further integration over the positive frequencies will yield a solution to the problem.

3. Solution

Using the obtained expressions (A3) and (A5) (from the appendix), we now seek the mean square values for the components of the electric strength of a thermal field at the boundary $z = 0$. To find a convenient form of solution, one should bear in mind the fact that in the integration over real variable λ, the variable q takes either pure imaginary or pure real values. As a result,

$$
\begin{aligned}
\langle|E_z|^2\rangle = {} & -\frac{2\Theta(\omega, T_1)}{\pi\omega\varepsilon}\mathrm{Re}\int_0^\infty \frac{\lambda^3 \mathrm{d}\lambda}{\mathrm{i}q}\left\{\frac{\left[\cosh(ql) + (\beta_2/\widetilde{\alpha}_2)\sinh(ql)\right]}{\widetilde{D}}\right\} \\
& + \frac{2\left[\Theta(\omega, T_1) - \Theta(\omega, T_2)\right]}{\pi\omega\varepsilon}\mathrm{Re}\int_0^\infty \frac{\lambda^3 \mathrm{d}\lambda}{\mathrm{i}q^*}\left\{\frac{\beta_2}{\widetilde{\alpha}_2}|\widetilde{D}|^{-2}\right\} \\
\langle|E_x|^2\rangle = \langle|E_y|^2\rangle = {} & -\frac{\Theta(\omega, T_1)}{\pi\omega\varepsilon}\mathrm{Re}\int_0^\infty \frac{\lambda \mathrm{d}\lambda}{\mathrm{i}}\left\{\frac{k^2}{q}\frac{\left[\cosh(ql) + (\beta_2/\alpha_2)\sinh(ql)\right]}{D}\right. \\
& \left. - q\frac{\beta_1}{\widetilde{\alpha}_1}\frac{\left[\sinh(ql) + (\beta_2/\widetilde{\alpha}_2)\cosh(ql)\right]}{\widetilde{D}}\right\} + \frac{\left[\Theta(\omega, T_1) - \Theta(\omega, T_2)\right]}{\pi\omega\varepsilon} \\
& \times\mathrm{Re}\int_0^\infty \frac{\lambda \mathrm{d}\lambda}{\mathrm{i}}\left\{\frac{k^2\beta_2^*}{q\alpha_2^*}|D|^{-2} - q\frac{\beta_2|\beta_1|^2}{\widetilde{\alpha}_2|\widetilde{\alpha}_1|^2}|\widetilde{D}|^{-2}\right\}.
\end{aligned} \tag{4}
$$

Rearranging D and $\widetilde{D}$, $\widetilde{\alpha}_j$ and α_j ($\mathrm{j} = 1, 2$) and vice versa in these formulae will yield the mean square values for the magnetic strength components of a fluctuating field. Note that at $T_1 = T_2$ the resulting expressions are exactly the same as the corresponding equations in the equilibrium problem [3, 4].

Knowing the mean square values of the fluctuating-field components, we can find from (2) the spectral density of the force acting on the lower medium:

$$
\begin{aligned}
\widetilde{F}_\omega = T_\omega^{zz} = {} & -\frac{\Theta(\omega, T_1)}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda \mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\Delta}{D} + \frac{\widetilde{\Delta}}{\widetilde{D}}\right) \\
& + \frac{\left[\Theta(\omega, T_2) - \Theta(\omega, T_1)\right]}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda \mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\delta_1}{|D|^2} + \frac{\widetilde{\delta}_1}{|\widetilde{D}|^2}\right)
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
\Delta &= \left(\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right)\sinh(ql) + \left(1 + \frac{\beta_1\beta_2}{\alpha_1\alpha_2}\right)\cosh(ql) \\
\widetilde{\Delta} &= \left(\frac{\beta_1}{\widetilde{\alpha}_1} + \frac{\beta_2}{\widetilde{\alpha}_2}\right)\sinh(ql) + \left(1 + \frac{\beta_1\beta_2}{\widetilde{\alpha}_1\widetilde{\alpha}_2}\right)\cosh(ql) \\
\delta_1 &= \frac{\beta_2}{\alpha_2}\left(\frac{q}{q^*} - \frac{|\beta_1|^2}{|\alpha_1|^2}\right) \qquad \widetilde{\delta}_1 = \frac{\beta_2}{\widetilde{\alpha}_2}\left(\frac{q}{q^*} - \frac{|\beta_1|^2}{|\widetilde{\alpha}_1|^2}\right).
\end{aligned}
$$

To represent (5) in several equivalent forms, use the following equalities that are easy to prove:

$$\begin{aligned}
&\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\Delta}{D}\right) = -\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\delta_1}{|D|^2}+\frac{\delta_2}{|D|^2}\right)\\
&\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\widetilde{\Delta}}{\widetilde{D}}\right) = -\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\widetilde{\delta}_1}{|\widetilde{D}|^2}+\frac{\widetilde{\delta}_2}{|\widetilde{D}|^2}\right)
\end{aligned} \tag{6}$$

where

$$\delta_2 = \frac{\beta_1}{\alpha_1}\left(\frac{q}{q^*}-\frac{|\beta_2|^2}{|\alpha_2|^2}\right) \qquad \widetilde{\delta}_2 = \frac{\beta_1}{\widetilde{\alpha}_1}\left(\frac{q}{q^*}-\frac{|\beta_2|^2}{|\widetilde{\alpha}_2|^2}\right).$$

Transformation of (5) with account for (6) will yield the following equivalent expressions for the spectral force density:

$$\widetilde{F}_\omega = \frac{\Theta(\omega,T_1)}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\delta_2}{|D|^2}+\frac{\widetilde{\delta}_2}{|\widetilde{D}|^2}\right) + \frac{\Theta(\omega,T_2)}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\delta_1}{|D|^2}+\frac{\widetilde{\delta}_1}{|\widetilde{D}|^2}\right) \tag{7}$$

$$\begin{aligned}
\widetilde{F}_\omega = &-\frac{\Theta(\omega,T_2)}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\Delta}{D}+\frac{\widetilde{\Delta}}{\widetilde{D}}\right)\\
&+\frac{[\Theta(\omega,T_1)-\Theta(\omega,T_2)]}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\delta_2}{|D|^2}+\frac{\widetilde{\delta}_2}{|\widetilde{D}|^2}\right)
\end{aligned} \tag{8}$$

$$\begin{aligned}
\widetilde{F}_\omega = &-\frac{[\Theta(\omega,T_1)+\Theta(\omega,T_2)]}{4\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\Delta}{D}+\frac{\widetilde{\Delta}}{\widetilde{D}}\right)\\
&+\frac{[\Theta(\omega,T_2)-\Theta(\omega,T_1)]}{4\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\delta_1-\delta_2}{|D|^2}+\frac{\widetilde{\delta}_1-\widetilde{\delta}_2}{|\widetilde{D}|^2}\right)
\end{aligned} \tag{9}$$

where it is seen that for identical materials we shall have

$$\widetilde{F}_\omega = -\frac{[\Theta(\omega,T_1)+\Theta(\omega,T_2)]}{4\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left(\frac{\Delta}{D}+\frac{\widetilde{\Delta}}{\widetilde{D}}\right). \tag{10}$$

From all of the above equations there follows an expression for the spectral density of the attractive force in equilibrium, where $T_1 = T_2$. In (10) let us separate out spectral density of the pressure caused only by the radiation modes of a fluctuating electromagnetic field ($0 \leqslant \lambda \leqslant k$):

$$P_\omega = \frac{[\Theta(\omega,T_1)+\Theta(\omega,T_2)]}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}} = \frac{[\Theta(\omega,T_1)+\Theta(\omega,T_2)]\,k^3}{6\pi^2\omega} \tag{11}$$

which is compensated for by the same pressure on the opposite side of the body. As a result, we obtain the spectral coupling force that depends only on a distance between the solids:

$$\begin{aligned}
F_\omega(l) = &-\frac{[\Theta(\omega,T_1)+\Theta(\omega,T_2)]}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\mathrm{d}\lambda}{\mathrm{i}}\left\{\left[\frac{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}{(\alpha_1-\beta_1)(\alpha_2-\beta_2)}\exp(2ql)-1\right]^{-1}\right.\\
&\left.+\left[\frac{(\widetilde{\alpha}_1+\beta_1)(\widetilde{\alpha}_2+\beta_2)}{(\widetilde{\alpha}_1-\beta_1)(\widetilde{\alpha}_2-\beta_2)}\exp(2ql)-1\right]^{-1}\right\}.
\end{aligned} \tag{12}$$

Integration of this expression over positive frequencies yields the sought-after formula for the attractive force between two semi-infinite solids with different temperatures. At the same time, we change the variables in this expression, assuming $q = \mathrm{i}kp$, $q_j = \mathrm{i}ks_j$ ($\mathrm{Re}\, q_j > 0$), where $s_j = \sqrt{(\alpha_j\widetilde{\alpha}_j - 1) + p^2}$, ($j = 1, 2$) to obtain

$$F = \frac{\hbar}{4\pi^2 c^3}\mathrm{Re}\int_0^\infty \int p^2\omega^3 \left[\coth\left(\frac{\hbar\omega}{2k_B T_1}\right) + \coth\left(\frac{\hbar\omega}{2k_B T_2}\right)\right] \times \left\{\left[\frac{(\alpha_1 p + s_1)(\alpha_2 p + s_2)}{(\alpha_1 p - s_1)(\alpha_2 p - s_2)}\exp(2\mathrm{i}kpl) - 1\right]^{-1} + \left[\frac{(\widetilde{\alpha}_1 p + s_1)(\widetilde{\alpha}_2 p + s_2)}{(\widetilde{\alpha}_1 p - s_1)(\widetilde{\alpha}_2 p - s_2)}\exp(2\mathrm{i}kpl) - 1\right]^{-1}\right\} \mathrm{d}p\, \mathrm{d}\omega \tag{13}$$

where p integration goes from 1 to 0 and then along the imaginary axis from 0 to $-\mathrm{i}\infty$. Note that the force is sought as the stress tensor projection on the unit external normal $\boldsymbol{n}$ to the surface $z = 0$ or on the external normal $\boldsymbol{n}'$ to the surface $z = l$, so it is obvious that the forces applied to these surfaces should differ in sign, $\boldsymbol{F}_{z=0} = -\boldsymbol{F}_{z=l}$.

The complex integral (13) is reduced to a form convenient for calculations by changing the integration paths in the planes of the complex variables ω and p, such as that performed in [3]. Specifically, p integration should be done only over real values (from 1 to ∞), while for ω only along the imaginary axis (from 0 to $-\mathrm{i}\infty$). Then the exponential function will always have a real index. We have to bear in mind that in our case $\varepsilon(\omega)$ is an analytical function in the lower half-plane of complex variable $\omega = \omega' + \mathrm{i}\omega''$ because of which we take an expansion in terms of $\exp(\mathrm{i}\omega t)$. We need to account for the fact that the functions $\coth(\hbar\omega/2k_B T_1)$ and $\coth(\hbar\omega/2k_B T_2)$ have an infinite number of poles on the imaginary axis, which are, respectively,

$$\omega_n = -\mathrm{i}\xi_n = -\mathrm{i}\frac{2\pi k_B T_1}{\hbar}n \qquad \omega_m = -\mathrm{i}\xi_m = -\mathrm{i}\frac{2\pi k_B T_2}{\hbar}m$$

where n and m are integers. When the ω-integration path shifts to the imaginary axis, these poles should be bypassed on a semicircle.

For convenience of comparison with the equilibrium case, we assume that both the solids are identical, nonmagnetic, with a vacuum gap between them. We then obtain

$$F = \frac{k_B T_1}{2\pi c^3}\sum_{n=0}^{\infty}{}' \int_1^\infty p^2\xi_n^3 \left\{\left[\left(\frac{s_n + p}{s_n - p}\right)^2 \exp(2p\xi_n l/c) - 1\right]^{-1} + \left[\left(\frac{s_n + \varepsilon_n p}{s_n - \varepsilon_n p}\right)^2 \exp(2p\xi_n l/c) - 1\right]^{-1}\right\} \mathrm{d}p + \frac{k_B T_2}{2\pi c^3}\sum_{m=0}^{\infty}{}' \int_1^\infty p^2\xi_m^3 \left\{\left[\left(\frac{s_m + p}{s_m - p}\right)^2 \exp(2p\xi_m l/c) - 1\right]^{-1} + \left[\left(\frac{s_m + \varepsilon_m p}{s_m - \varepsilon_m p}\right)^2 \exp(2p\xi_m l/c) - 1\right]^{-1}\right\} \mathrm{d}p \tag{14}$$

where $\varepsilon_n = \varepsilon(\mathrm{i}\xi_n)$, $\varepsilon_m = \varepsilon(\mathrm{i}\xi_m)$ are the values of the dielectric constants on the imaginary axis, $s_n = \sqrt{\varepsilon_n - 1 + p^2}$, $s_m = \sqrt{\varepsilon_m - 1 + p^2}$. The prime in the sum indicates that all terms with $n = 0$ and $m = 0$ have to be taken at half-weight.

4. Discussion

Formula (14) can be used to find the force F for any distance and any relationship of temperatures T_1 and T_2. At $T_1 = T_2$, formula (14) fully coincides with the result obtained for the first time in [3]. Analysis of the expression for the attractive force is based on the fact that the key role in the sums is that of the terms in which $\xi_n \sim c/l$ and $\xi_m \sim c/l$, or the terms with the numbers $n \sim c\hbar/lk_B T_1$ and $m \sim c\hbar/lk_B T_2$. The situations $lk_B T/c\hbar \ll 1$ and $lk_B T/c\hbar \gg 1$, where $T_1 = T_2 = T$, have been analysed in detail in [3, 5]. It follows therefrom that at small distances the temperature of the bodies is absolutely unimportant, and $F \sim l^{-3}$, if $l \ll c\hbar/k_B T$ and $l \ll \lambda_0$, where λ_0 is the characteristic wavelength in the absorption spectrum from the object of interest. If $l \ll c\hbar/k_B T$, but $l \gg \lambda_0$, we have: $F \sim l^{-4}$. In particular for metals, we have $F = \hbar c\pi^2/240l^4$ regardless of the kind of metal. At $l \gg c\hbar/k_B T$, $F \simeq k_B T(\varepsilon_0 - 1)^2/8\pi l^3(\varepsilon_0 + 1)^2$ and it depends on the static values of the dielectric constant ε_0 of a material.

When $T_1 \neq T_2$, the relationships may have a wider variety of forms:

$$
\begin{aligned}
&l \ll \frac{c\hbar}{k_B T_1} \qquad l \ll \frac{c\hbar}{k_B T_2} \qquad l \ll \lambda_0 \qquad F \simeq \frac{\hbar}{8\pi^2 l^3}\int_0^\infty \left[\frac{\varepsilon(\mathrm{i}\xi) - 1}{\varepsilon(\mathrm{i}\xi) + 1}\right]^2 \mathrm{d}\xi \\
&l \ll \frac{c\hbar}{k_B T_1} \qquad l \ll \frac{c\hbar}{k_B T_2} \qquad l \gg \lambda_0 \qquad F \simeq \frac{\pi^2 \hbar c}{240 l^4} \\
&l \gg \frac{c\hbar}{k_B T_1} \qquad l \gg \frac{c\hbar}{k_B T_2} \qquad F \simeq \frac{k_B(T_1 + T_2)}{16\pi l^3}\left(\frac{\varepsilon_0 - 1}{\varepsilon_0 + 1}\right)^2 \\
&l \gg \frac{c\hbar}{k_B T_1} \qquad l \ll \frac{c\hbar}{k_B T_2} \qquad l \ll \lambda_0 \qquad F \simeq \frac{\hbar}{16\pi^2 l^3}\int_0^\infty \left[\frac{\varepsilon(\mathrm{i}\xi) - 1}{\varepsilon(\mathrm{i}\xi) + 1}\right]^2 \mathrm{d}\xi \\
&\qquad\qquad + \frac{k_B T_1}{16\pi l^3}\left(\frac{\varepsilon_0 - 1}{\varepsilon_0 + 1}\right)^2 \\
&l \gg \frac{c\hbar}{k_B T_1} \qquad l \ll \frac{c\hbar}{k_B T_2} \qquad l \gg \lambda_0 \qquad F \simeq \frac{k_B T_1}{16\pi l^3}\left(\frac{\varepsilon_0 - 1}{\varepsilon_0 + 1}\right)^2 + \frac{\pi^2 \hbar c}{480 l^4}.
\end{aligned}
\tag{15}
$$

For numerical computation at any distance l and any temperatures of solids it is more convenient to use another form of solution. After the substitution $p = x/n$ and $p = x/m$ the solution (14) can be represented as

$$
\begin{aligned}
F = {} & \frac{4\pi^2 k_B{}^4 T_1{}^4}{\hbar^3 c^3} {\sum_{n=0}^{\infty}}' \int_n^\infty x^2 \left\{ \left[\left(\frac{ns_n + x}{ns_n - x}\right)^2 \exp(4\pi T_1 x l/\hbar c) - 1 \right]^{-1} \right. \\
& \left. + \left[\left(\frac{ns_n + \varepsilon_n x}{ns_n - \varepsilon_n x}\right)^2 \exp(4\pi T_1 x l/\hbar c) - 1 \right]^{-1} \right\} \mathrm{d}x \\
& + \frac{4\pi^2 k_B{}^4 T_2{}^4}{\hbar^3 c^3} {\sum_{m=0}^{\infty}}' \int_m^\infty x^2 \left\{ \left[\left(\frac{ms_m + x}{ms_m - x}\right)^2 \exp(4\pi T_2 x l/\hbar c) - 1 \right]^{-1} \right. \\
& \left. + \left[\left(\frac{ms_m + \varepsilon_m x}{ms_m - \varepsilon_m x}\right)^2 \exp(4\pi T_2 x l/c) - 1 \right]^{-1} \right\} \mathrm{d}x
\end{aligned}
\tag{16}
$$

where $ns_n = \sqrt{n^2(\varepsilon_n - 1) + x^2}$, $ms_m = \sqrt{m^2(\varepsilon_m - 1) + x^2}$.

The order of magnitude of the attractive force is provided here for the distances $l = 10$–100 Å, typical for probe microscopy. In this case, $l \ll \hbar c/k_B T_{1,2}$ and $l \ll \lambda_0$ for $T_{1,2} = 300$–2000 K, hence, $F \simeq \hbar\overline{\omega}/8\pi^2 l^3$, where $\overline{\omega}$ is the characteristic frequency in the absorption spectrum of materials. For estimation, assume $\overline{\omega} \sim 10^{14}$ rad s^{-1}, then $F \sim 10^2$–10^4 N m^{-2}, which is by far (by many orders) lower than both the destruction threshold and the yield stress of solid matter.

As shown in [5], the interaction force between two bodies with dielectric constants ε_1 and ε_2, separated by a gap of width l filled with an absorbing medium $\varepsilon_3 = \varepsilon_3' + \mathrm{i}\varepsilon_3''$, can be obtained from the expression for the interaction force of two bodies separated by vacuum, if one multiplies either term in the sums (14) at $T_1 = T_2$ by $\varepsilon_3^{3/2}$, and replaces ε_1 and ε_2 in all the terms by $\varepsilon_1/\varepsilon_3$ and $\varepsilon_2/\varepsilon_3$, and l by $l\sqrt{\varepsilon_3}$, respectively. Obviously, this can only be done in the equilibrium case, as in the derivation of the stress tensor in an absorbing medium the authors in [5] took into account the constancy of the chemical potential throughout the thickness of a film filling the gap between the bodies, which is impossible given a temperature gradient.

5. Conclusion

In this paper, an expression for the attractive force between two solids that, in a general case, may differ in temperature, is obtained. The solution is found by seeking thermal losses of the field of a point dipole placed in the gap between the solids and by using the generalized Kirchhoff's law. Transformation of complex integrals was used to obtain a variety of forms of the expression for the force spectral density. Options of the solution are considered depending on a relation between the temperatures of these solids. It is shown that, if the bodies have the same temperature, the obtained expression for the coupling force between two solid media yields a formula corresponding to the equilibrium case.

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Appendix. Solution of the regular problem

The losses of the diffraction field $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ of the point dipoles in half-spaces $z \leqslant 0$ and $z \geqslant l$ are sought as the energy flow into absorbing material, i.e. as the integral of the Poynting vector over the planes $z = 0$ and $z = l$, respectively. For example,

$$Q^{(1)} = -\frac{c}{16\pi}\int_0^\infty r\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\varphi\{[\boldsymbol{E}_0, \boldsymbol{H}_0^*]_z + \mathrm{c.c.}\}_{z=0}. \tag{A1}$$

To this end, we define the dipole field in each of the three media. A common approach to solving the Maxwell equations in an inhomogeneous medium for the specified sources can be found in [12]. A solution of the boundary-value problem on the dipole field in a gap between two half-spaces was obtained in [4]. We are interested in the diffraction field in absorbing materials 1 and 2, therefore, we shall seek a complete solution to this problem and extend the result to the cases of three regions with the corresponding boundary conditions in the planes $z = 0$ and $z = l$. The solution will be found by analogy with the solving of the problem on a dipole above the conducting ground [13, 14]; in all of the three media we determine the Hertz vector $\boldsymbol{Z}$. It enters ordinary relationships with the scalar and vector potentials,

$$\varphi = -\frac{1}{\varepsilon\mu}\mathrm{div}\boldsymbol{Z} \qquad \boldsymbol{A} = \frac{1}{c}\frac{\partial \boldsymbol{Z}}{\partial t}.$$

Using the Lorentz condition and the expressions via the potentials φ and $\boldsymbol{A}$, we obtain the relation between $\boldsymbol{E}_0$, $\boldsymbol{H}_0$ and $\boldsymbol{Z}$. For example, in absorbing media we shall have

$$\begin{aligned} \boldsymbol{E}_0^{(j)} &= \frac{1}{\varepsilon_j\mu_j}\left\{\mathrm{grad}(\mathrm{div}\boldsymbol{Z}^{(j)}) + k_j^2\boldsymbol{Z}^{(j)}\right\} \\ \boldsymbol{H}_0^{(j)} &= \frac{\mathrm{i}k_0}{\mu_j}\mathrm{rot}\boldsymbol{Z}^{(j)} \end{aligned} \tag{A2}$$

where $k_0 = \omega/c$ is the wavenumber in vacuum, $k_j^2 = k_0^2\varepsilon_j\mu_j$, $j = 1, 2$, and it is assumed that $Z \sim \mathrm{e}^{\mathrm{i}\omega t}$. By similar formulae one can find the field in the gap, where $k^2 = k_0^2\varepsilon\mu$.

The Maxwell equations in absorbing media and the gap can be met, if the Hertz vector is known for each of the three media. In our case, to find the Cartesian components of the Hertz vector in the gap and the absorbing materials we need to solve the equations

$$\Delta\boldsymbol{Z} + k^2\boldsymbol{Z} = -4\pi\mu\boldsymbol{p}\delta(\boldsymbol{r} - \boldsymbol{r}')$$

and

$$\Delta\boldsymbol{Z}^{(j)} + k_j^2\boldsymbol{Z}^{(j)} = 0 \qquad (j = 1, 2) \tag{A3}$$

where $\delta(\boldsymbol{r} - \boldsymbol{r}')$ is the delta function, $\boldsymbol{r}$ and $\boldsymbol{r}'$ are the observation point and the point dipole coordinates, respectively. If the point dipole with the moment $p = (0, 0, p_z)$ is oriented in the $0z$-axis, then the equations are satisfied given $Z = (0, 0, Z_z)$. For a horizontal orientation of the dipole in the $0x$-or $0y$-axes, as shown in [13], one needs to assume—to avoid contradiction in the boundary conditions—that it is the vertical component of the Hertz vector that is induced, i.e. $\boldsymbol{Z} = (Z_x, 0, \widetilde{Z}_z)$ and $\boldsymbol{Z} = (Z_, 0, \widetilde{Z}_z)$. Physically, this is related to the effects of media 1 and 2. In other words, one more field $\widetilde{Z}$ is created by the secondary sources in media 1 and 2, which is the solution to the homogeneous equations (A3). The latter have to be completed with the boundary conditions expressing equality between the tangent components of the diffraction field at the boundaries $z = 0$ and $z = 1$. For the z-oriented dipole, when $\boldsymbol{Z} = (0, 0, Z_z)$, we have

$$\frac{Z_z}{\mu} = \frac{Z_z^{(j)}}{\mu_j} \qquad \frac{1}{\varepsilon\mu}\frac{\partial Z_z}{\partial z} = \frac{1}{\varepsilon_j\mu_j}\frac{\partial Z_z^{(j)}}{\partial z}$$

for the x-oriented dipole, when $Z = (Z_x, 0, \widetilde{Z}_z)$

$$\begin{aligned} &Z_x = Z_x^{(j)} \qquad \frac{1}{\mu}\frac{\partial Z_x}{\partial z} = \frac{1}{\mu_j}\frac{\partial Z_x^{(j)}}{\partial z} \qquad \frac{\widetilde{Z}_z}{\mu} = \frac{\widetilde{Z}_z^{(j)}}{\mu_j} \\ &\frac{1}{\varepsilon\mu}\left(\frac{\partial\widetilde{Z}_z}{\partial z} + \frac{\partial Z_x}{\partial x}\right) = \frac{1}{\varepsilon_j\mu_j}\left(\frac{\partial\widetilde{Z}_z^{(j)}}{\partial z} + \frac{\partial Z_x^{(j)}}{\partial x}\right) \qquad (j = 1, 2). \end{aligned} \tag{A4}$$

Similar conditions at the boundary are obtained for the y-oriented dipole.

With account for the form of equations (A3), we seek the solution in the following way. Assume that in the gap between the absorbing media $Z_z = p_z Z_v$; $\widetilde{Z}_z = p_x \cos\varphi\widetilde{Z}_v$;

$\widetilde{Z}_z = p_y \sin\varphi \widetilde{Z}_v$; $Z_x = p_x Z_h$; $Z_y = p_y Z_h$ where

$$Z_v = \mu \int_0^\infty J_0(\lambda r) \exp(-q|z-h|) \frac{\lambda \mathrm{d}\lambda}{q} + \int_0^\infty J_0(\lambda r)[G_- \exp(-qz) + G_+ \exp(qz)] \,\mathrm{d}\lambda$$

$$Z_h = \mu \int_0^\infty J_0(\lambda r) \exp(-q|z-h|) \frac{\lambda \mathrm{d}\lambda}{q} + \int_0^\infty J_0(\lambda r)[F_- \exp(-qz) + F_+ \exp(qz)] \,\mathrm{d}\lambda$$

$$\widetilde{Z}_v = \int_0^\infty J_1(\lambda r)[H_- \exp(-qz) + H_+ \exp(qz)] \,\mathrm{d}\lambda$$

in absorbing materials: $Z_z^{(j)} = p_z Z_v^{(j)}$; $\widetilde{Z}_z^{(j)} = p_x \cos\varphi \widetilde{Z}_v^{(j)}$; $\widetilde{Z}_z^{(j)} = p_y \sin\varphi \widetilde{Z}_v^{(j)}$; $Z_x^{(j)} = p_x Z_h^{(j)}$; $Z_y^{(j)} = p_y Z_h^{(j)}$ where

$$Z_v^{(1)} = \int_0^\infty J_0(\lambda r) G_1 \exp(q_1 z) \,\mathrm{d}\lambda; \qquad Z_h^{(1)} = \int_0^\infty J_0(\lambda r) F_1 \exp(q_1 z) \,\mathrm{d}\lambda$$

$$\widetilde{Z}_v^{(1)} = \int_0^\infty J_1(\lambda r) H_1 \exp(q_1 z) \,\mathrm{d}\lambda \qquad Z_v^{(2)} = \int_0^\infty J_0(\lambda r) G_2 \exp(-q_2(z-l)) \,\mathrm{d}\lambda$$

$$Z_h^{(2)} = \int_0^\infty J_0(\lambda r) F_2 \exp(-q_2(z-l)) \,\mathrm{d}\lambda \qquad \widetilde{Z}_v^{(2)} = \int_0^\infty J_1(\lambda r) H_2 \exp(-q_2(z-l)) \,\mathrm{d}\lambda$$

where J_n is the Bessel function of order n, $q = \sqrt{\lambda^2 - k^2}$, $q_j = \sqrt{\lambda^2 - k_j^2} (j = 1, 2)$. From the boundary conditions (A4) we find the equations to define the coefficients $G_\pm$, $F_\pm$, $H_\pm$, G_j, F_j, H_j, $(j = 1, 2)$:

$$G_1 = \alpha_1 \left[\mu \frac{\lambda}{q} \exp(-qh) + G_+ + G_- \right]$$

$$G_1 = \frac{\gamma_1}{\beta_1} \left[\mu \frac{\lambda}{q} \exp(-qh) + G_+ - G_- \right]$$

$$F_1 = \mu \frac{\lambda}{q} \exp(-qh) + F_+ + F_-$$

$$F_1 = \frac{\alpha_1}{\beta_1} \left[\mu \frac{\lambda}{q} \exp(-qh) + F_+ - F_- \right]$$

$$H_1 = \alpha_1 (H_+ + H_-)$$

$$\lambda F_1 - q_1 H_1 = \gamma_1 \left[\lambda \left(\mu \frac{\lambda}{q} \exp(-qh) + F_+ + F_- \right) + q(H_- - H_+) \right]$$

$$G_2 = \alpha_2 \left[\mu \frac{\lambda}{q} \exp(-q(l-h)) + G_+ \exp(ql) + G_- \exp(-ql) \right]$$

$$G_2 = \frac{\gamma_2}{\beta_2} \left[G_+ \exp(ql) - G_- \exp(-ql) - \mu \frac{\lambda}{q} \exp(-q(l-h)) \right]$$

$$F_2 = \mu \frac{\lambda}{q} \exp(-q(l-h)) + F_+ \exp(ql) + F_- \exp(-ql)$$

$$F_2 = \frac{\alpha_2}{\beta_2} \left[\mu \frac{\lambda}{q} \exp(-q(l-h)) - F_+ \exp(ql) + F_- \exp(-ql) \right]$$

$$H_2 = \alpha_2 [H_+ \exp(ql) + H_- \exp(-ql)]$$

$$\lambda F_2 + q_2 H_2 = \gamma_2 \Big[\lambda \left(\mu \frac{\lambda}{q} \exp(-q(l-h)) + F_+ \exp(ql) + F_- \exp(-ql) \right) + q\,(H_- \exp(-ql) - H_+ \exp(ql)) \Big]$$

where

$$\beta_j = \frac{q_j}{q} \qquad \gamma_j = \frac{k_j^2}{k^2} = \frac{\varepsilon_j \mu_j}{\varepsilon \mu} = \alpha_j \tilde{\alpha}_j \qquad \alpha_j = \frac{\mu_j}{\mu} \qquad \tilde{\alpha}_j = \frac{\varepsilon_j}{\varepsilon} \qquad (j = 1, 2).$$

Solving of this set yields:

$$G_1 = 2\mu_1 \frac{\lambda}{q} \left[\cosh(ql - qh) + \frac{\beta_2}{\widetilde{\alpha}_2} \sinh(ql - qh) \right] \widetilde{D}^{-1}$$

$$G_2 = 2\mu_2 \frac{\lambda}{q} \left[\cosh(qh) + \frac{\beta_1}{\widetilde{\alpha}_1} \sinh(qh) \right] \widetilde{D}^{-1}$$

$$F_1 = 2\mu \frac{\lambda}{q} \left[\cosh(ql - qh) + \frac{\beta_2}{\alpha_2} \sinh(ql - qh) \right] D^{-1}$$

$$F_2 = 2\mu \frac{\lambda}{q} \left[\cosh(qh) + \frac{\beta_1}{\alpha_1} \sinh(qh) \right] D^{-1}$$

$$H_1 = \alpha_1 \frac{\lambda}{q} \left\{ F_1 \left(\frac{1 - \gamma_1}{\gamma_1} \right) \left[\cosh(ql) + \frac{\beta_2}{\widetilde{\alpha}_2} \sinh(ql) \right] - F_2 \left(\frac{1 - \gamma_2}{\gamma_2} \right) \right\} \widetilde{D}^{-1}$$

$$H_2 = \alpha_2 \frac{\lambda}{q} \left\{ F_1 \left(\frac{1 - \gamma_1}{\gamma_1} \right) - F_2 \left(\frac{1 - \gamma_2}{\gamma_2} \right) \left[\cosh(ql) + \frac{\beta_1}{\widetilde{\alpha}_1} \sinh(ql) \right] \right\} \widetilde{D}^{-1}$$

where

$$D = \left(\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right) \cosh(ql) + \left(1 + \frac{\beta_1 \beta_2}{\alpha_1 \alpha_2} \right) \sinh(ql)$$

$$\widetilde{D} = \left(\frac{\beta_1}{\widetilde{\alpha}_1} + \frac{\beta_2}{\widetilde{\alpha}_2} \right) \cosh(ql) + \left(1 + \frac{\beta_1 \beta_2}{\widetilde{\alpha}_1 \widetilde{\alpha}_2} \right) \sinh(ql).$$

It is more convenient to search for the fields in the cylindrical-coordinate system, where the Hertz vector components are related with the Cartesian ones as

$$Z_r = (p_x \cos\varphi + p_y \sin\varphi) Z_h$$
$$Z_\varphi = (p_y \cos\varphi - p_x \sin\varphi) Z_h$$
$$Z_z = p_z Z_v + (p_x \cos\varphi + p_y \sin\varphi) \widetilde{Z}_v.$$

Calculations of losses by formula (A1), which involved the recurrent relations between the Bessel functions and the properties of the delta function, lead to the following result for the case $h = 0$:

$$Q_z^{(1)} = \frac{1}{\omega\varepsilon} \mathrm{Re} \int_0^\infty \frac{\lambda^3 \mathrm{d}\lambda}{\mathrm{i}|q|^2} \left\{ \frac{q_1}{\widetilde{\alpha}_1} \frac{|\cosh(ql) + (\beta_2/\widetilde{\alpha}_2)\sinh(ql)|^2}{|\widetilde{D}|^2} \right\}$$

$$Q_z^{(2)} = \frac{1}{\omega\varepsilon} \mathrm{Re} \int_0^\infty \frac{\lambda^3 \mathrm{d}\lambda}{\mathrm{i}|q|^2} \left\{ \frac{q_2}{\widetilde{\alpha}_2} \frac{1}{|\widetilde{D}|^2} \right\}$$

$$Q_x^{(1)} = \frac{1}{2\omega\varepsilon} \mathrm{Re} \int_0^\infty \frac{\lambda \mathrm{d}\lambda}{\mathrm{i}q} \left\{ q^2 \frac{\beta_1}{\widetilde{\alpha}_1} \frac{|\sinh(ql) + (\beta_2/\widetilde{\alpha}_2)\cosh(ql)|^2}{|\widetilde{D}|^2} - k^2 \frac{\beta_1^*}{\alpha_1^*} \frac{|\cosh(ql) + (\beta_2/\alpha_2)\sinh(ql)|^2}{|D|^2} \right\} \tag{A5}$$

$$Q_x^{(2)} = \frac{1}{2\omega\varepsilon} \mathrm{Re} \int_0^\infty \frac{\lambda \mathrm{d}\lambda}{\mathrm{i}q} \left\{ q^2 \frac{\beta_2}{\widetilde{\alpha}_2} \frac{|\beta_1|^2}{|\widetilde{\alpha}_1|^2} \frac{1}{|\widetilde{D}|^2} - k^2 \frac{\beta_2^*}{\alpha_2^*} \frac{1}{|D|^2} \right\}.$$

Note that $Q_y^{(1)} = Q_x^{(1)}$ and $Q_y^{(2)} = Q_x^{(2)}$. Naturally, in the limiting case $l \to \infty$ one easily finds the corresponding formulae for the dipole field losses over conducting surface.

Computation of losses incurred in absorbing media by the diffraction field of a point dipole oriented in different axes completes the regular part of the problem.

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